Bäcklund transformations for nonlinear evolution equations: Hilbert space approach

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Abstract

A new method of determining Bäcklund transformations for nonlinear partial differential equations of the evolution type is introduced. Using the Hilbert space approach the problem of finding Bäcklund transformations is brought down to the solution of an abstract equation in Hilbert space.

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1. Introduction

In recent years, the interest in the Bäcklund transformations is steadily increasing in connection with a general increase in the understanding of methods for solution of nonlinear partial differential equations. Let us only recall the rôle played by the Miura transformation in discovery of the method of inverse scattering [1].

In the recent paper [2] we introduced a new method for finding linearization transformations for nonlinear partial differential equations of the evolution type based on the Hilbert space approach to the theory of nonlinear dynamical systems developed by the author [2-7]. The theory was illustrated by an example of the Burgers equation (we obtained in a simple way the celebrated Hopf-Cole transformation). The present work is devoted to a generalization of the treatment to the case involving general Bäcklund transformations. Proceeding analogously as in the paper [2] we demonstrate that the problem of finding Bäcklund transformations for evolution equations can be reduced to the solution of an abstract equation in Hilbert space. We illustrate the algorithm on the example of the Miura transformation.

2. Bäcklund transformations

We begin with a brief account of the Hilbert space description of nonlinear partial differential equations of the evolution type [4]. Consider the equation

$$\partial_t u(x,t) = F(u, D^\alpha u), \qquad u(x,0) = u_0(x), \tag{1}$$

where $u : \mathbf{R}^s \times \mathbf{R} \to \mathbf{R}$, $D^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \ldots \partial x_s^{\beta_s}$, $|\beta| = \sum_{i=1}^{s} \beta_i$, F is analytic in u, $D^\alpha u$ and $u_0 \in L^2_{\mathbf{R}}(\mathbf{R}^s, d^s x)$ (real Hilbert space of square-integrable functions).

(1) and we assume that u is square-integrable. We define the vectors $|u,t\rangle$ as follows

$$|u,t\rangle = \exp\left[\frac{1}{2}\left(\int d^s x u^2 - \int d^s x u_0^2\right)\right]|u\rangle. \tag{2}$$

Suppose now we are given a boson operator of the form

$$M = \int d^s x a^\dagger(x) F(a(x), D^\alpha a(x)), \tag{3}$$

where $a^\dagger(x)$ and $a(x)$ are the standard Bose field operators.

An easy differentiation shows that the vectors (2) satisfy the following linear evolution equation in Hilbert space

$$\frac{d}{dt}|u,t\rangle = M|u,t\rangle, \qquad |u,0\rangle = |u_0\rangle. \tag{4}$$

Taking into account (2) we find that the following eigenvalue equation holds true

$$a(x)|u_0,t\rangle = u[u_0|x,t]|u_0,t\rangle, \tag{5}$$

where $|u_0,t\rangle$ is the solution of (4) and $u[u_0|x,t]$ is the solution of (1) (the square brackets designate the functional dependence of u on u_0).

It thus appears that the nonlinear equation (1) can be brought down to the linear abstract Schrödinger-like equation (4). The restriction to square integrable data is not too serious. Indeed, the approach was shown to work also in the case when the initial data were not square integrable [2,4]. The postulate that the solutions are square integrable at any time is rather restrictive one. It should be noted, however, that there exist numerous equations of classical and of current interest satisfying nonlinear Schrödinger equation and Kadomtsev-Petviashvili equation. We should also mention that the treatment can be immediately extended to the case of complex multidimensional systems of partial differential equations (1) with a right-hand side dependent on x, t [4].

We now discuss the transformation of variables within the Hilbert space approach. Consider the following transformation

$$u' = \phi[u|x], \tag{6}$$

where ϕ is analytic in u.

Taking into account (2) we find that under (6) the "Hamiltonian" M transforms as

$$M' = \int d^s x a^\dagger(x)[\phi[a|x], M]. \tag{7}$$

Therefore, whenever the transformation (6) converts the equation (1) into the equation

$$\partial_t u' = F'(u', D^\beta u') \tag{8}$$

then the following commutation relation holds

$$[\phi[a|x], M] = F'(\phi[a|x], D^\beta \phi[a|x]). \tag{9}$$

On taking the Hermitian conjugate of (9) and using (B.7) we arrive at the following equation

$$M^\dagger|\phi(x)\rangle = F'(\phi[a^\dagger|x], D^\beta \phi[a^\dagger|x])|0\rangle, \tag{10}$$

where $|\phi(x)\rangle = \phi[a^\dagger|x]|0\rangle$.

The vector $|\phi(x)\rangle$ is related to the Bäcklund transformation (6) by

$$\phi[u|x] = \langle u|\phi(x)\rangle \exp\left(\frac{1}{2}\int d^s x u^2\right). \tag{11}$$

It thus appears that the problem of determining Bäcklund transformation (6) is equivalent to solving Hilbert space equation (10). The particular case

$$F'(u', D^\beta u') = Lu', \tag{12}$$

where L is a linear differential operator,

when (6) is the linearization transformation and (10) takes the form

$$M^\dagger|\phi(x)\rangle = L|\phi(x)\rangle \tag{13}$$

was discussed in ref. 2. Solving (13) we obtained in a simple way the celebrated Hopf-Cole transformation reducing the Burgers equation to the heat equation. We note that the case of the linearization transformation is the only one when (10) is linear. Nevertheless, it appears that there exist nontrivial cases when the Bäcklund transformations can be determined easily by solving (10). We now illustrate this observation by the example of the Miura transformation.

Example. Consider the modified Korteweg-de Vries equation

$$\partial_t u = -\partial_x^3 u + 6u^2 \partial_x u \tag{14}$$

and the Korteweg-de Vries equation

$$\partial_t u' = -\partial_x^3 u' + 6u' \partial_x u'. \tag{15}$$

We seek for the Bäcklund transformation ϕ such that

$$u' = \phi[u|x]. \tag{16}$$

Eq. [10] corresponding to (16) can be written as

$$M^\dagger |\phi(x)\rangle = -\partial_x^3 |\phi(x)\rangle + 6\phi[a^\dagger|x]\partial_x |\phi(x)\rangle, \tag{17}$$

where the conjugation $M^\dagger$ of the "Hamiltonian" corresponding to (14) is

$$M^\dagger = \int dx(-a^{\dagger\prime\prime\prime}(x) + 6a^{\dagger 2}(x)a^{\dagger\prime}(x))a(x). \tag{18}$$

On writing (18) in the coordinate representation (see appendix A) we obtain

$$\partial_{x_1}^3 \phi_1(x;x_1) = -\partial_x^3 \phi_1(x;x_1), \tag{19a}$$

$$(\partial_{x_1}^3 + \partial_{x_2}^3)\phi_2(x;x_1,x_2) = -\partial_x^3 \phi_2(x;x_1,x_2)$$

$$+ 6\phi_1(x;x_2)\partial_x \phi_1(x;x_1) + 6\phi_1(x;x_1)\partial_x \phi_1(x;x_2), \tag{19b}$$

$$
\begin{aligned}
&\sum_{i=1}^{n+2} \partial_{x_i}^3 \phi_{n+2}(x; x_1, \ldots, x_{n+2}) \\
&- 12 \sum_{i=1}^{n+2} \sum_{\substack{r,s\neq i \\ r>s}} \partial_{x_i}[\delta(x_i - x_r)\delta(x_i - x_s)\phi_n(x; x_1, \ldots, \check{x}_r, \ldots, \check{x}_s, \ldots, x_{n+2})] \\
&= -\partial_x^3 \phi_{n+2}(x; x_1, \ldots, x_{n+2}) \\
&+ 6 \sum_{r=1}^{n+1} \frac{1}{r!} \sum_{i_1=1}^{n+2} \sum_{i_2\neq i_1} \cdots \sum_{i_r\neq i_{r-1}} \phi_{n+2-r}(x; x_1, \ldots, \check{x}_{i_1}, \ldots, \check{x}_{i_r}, \ldots, x_{n+2}) \\
&\times \partial_x \phi_r(x; x_{i_1}, \ldots, x_{i_r}), \qquad n = 1, 2, \ldots, \infty,
\end{aligned}
\tag{19c}
$$

where $\phi_n(x; x_1, \ldots, x_n) = \langle x_1, \ldots, x_n | \phi(x) \rangle$ and the reversed hat over x_r, x_s and x_{i_1}, x_{i_r} denotes that these variables should be omitted from the set $\{x_1, \ldots, x_{n+2}\}$.

Hence, passing to the Fourier transformation we get

$$
(k^3 + k_1^3)\tilde{\phi}_1(k; k_1) = 0, \tag{20a}
$$

$$
(k^3 + k_1^3 + k_2^3)\tilde{\phi}_2(k; k_1, k_2) = -6k \int dk' \tilde{\phi}_1(k - k'; k_1)\tilde{\phi}_1(k', k_2), \tag{20b}
$$

$$
\begin{aligned}
&\left(k^3 + \sum_{i=1}^{n+2} k_i^3\right) \tilde{\phi}_{n+2}(k; k_1, \ldots, k_{n+2}) \\
&+ 12 \sum_{i=1}^{n+2} \sum_{\substack{r,s\neq i \\ r>s}} k_i \tilde{\phi}_n(k; k_1, \ldots, \check{k}_r, \ldots, \check{k}_s, \ldots, k_{n+2})_{|k_i \to k_i + k_r + k_s} \\
&= -6 \sum_{r=1}^{n+1} \frac{1}{r!} \sum_{i_1=1}^{n+2} \sum_{i_2\neq i_1} \cdots \sum_{i_r\neq i_{r-1}} \int dk' \tilde{\phi}_{n+2-r}(k - k'; k_1, \ldots, \check{k}_{i_1}, \ldots, \check{k}_{i_r}, \ldots, k_{n+2}) \\
&\times k' \tilde{\phi}_r(k'; k_{i_1}, \ldots, k_{i_r}), \qquad n = 1, 2, \ldots, \infty.
\end{aligned}
\tag{20c}
$$

Making use of the identities

$$(k^3 + k_1^3)\delta(k + k_1) = 0, \tag{21a}$$

$$\sum_{i=0}^{n} k_i^3 \delta\left(\sum_{i=0}^{n} k_i\right) = 3 \sum_{q>r>s} k_q k_r k_s \delta\left(\sum_{i=0}^{n} k_i\right), \tag{21b}$$

where $q, r, s \in \{0, 1, \ldots, n\}$, $n \geq 2$ and we set $k_0 = k$, one finds easily the following solution to (20)

$$\tilde{\phi}_1 = -ik\delta(k + k_1), \qquad \tilde{\phi}_2 = 2\delta(k + k_1 + k_2), \qquad \tilde{\phi}_n = 0, \qquad n \geq 3. \tag{22}$$

On performing Fourier's inverse transformations to (22) and using

$$|\phi(x)\rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \phi_n(x; x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle, \tag{23}$$

we obtain the solution to (17) of the form

$$|\phi(x)\rangle = |xx\rangle + \partial_x |x\rangle. \tag{24}$$

Hence taking into account (11) and (B.5) we finally arrive at the desired Bäcklund transformation (16) such that

$$u' = u^2 + \partial_x u. \tag{25}$$

The mapping (25) coincides with the celebrated Miura transformation relating solutions of the modified Korteweg-de Vries equation to solutions of the Korteweg-de Vries equation.

Conclusion

Applying the Hilbert space appproach to the theory of nonlinear dynamical systems developed by the author a new method is introduced in this work of finding Bäcklund transformations for nonlinear evolution equations. It should be noted that regardless of the form of eqs. (20b) and (20c) we have rederived the Miura transformation from (20) in purely algebraic mannner (we need not have solved any integral equation). The algorithm described herein is an example of the following general technique of the study of nonlinear partial differential equations based on the Hilbert space formalism. Namely, using the Hilbert space approach we first derive an abstract equation in Hilbert space corresponding to the considered nonlinear evolution problem. Then writing this equation in the coordinate representation and performing a Fourier transformation we obtain a system of algebraic equations related to the original problem. This technique was succesfully applied for finding first integrals [4,6] and linearization transformations [2] for nonlinear partial differential equations of the evolution type. The simplicity of the algorithm for determining Bäcklund transformations described in this work suggests that it would also be a useful tool in the study of nonlinear evolution equations.

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Appendix A. Coordinate representation

We first recall the basic properties of the coordinate representation. The Bose creation ($a^\dagger(x)$) and annihilation ($a(x)$) operators obey the canonical commutation relations

$$
\begin{aligned}
&[a(x), a^\dagger(x')] = \delta(x - x'), \\
&[a(x), a(x')] = [a^\dagger(x), a^\dagger(x')] = 0, \qquad x, x' \in \mathbf{R}^s.
\end{aligned}
\tag{A.1}
$$

Let us assume that there exists in a Hilbert space of states $\mathcal{H}$ where act Bose operators a unique normalized vector $|0\rangle$ (vacuum vector) such that

$$
a(x)|0\rangle = 0, \qquad \text{for every } x \in \mathbf{R}^s. \tag{A.2}
$$

We also assume that there is no nontrivial closed subspace of $\mathcal{H}$ which is invariant under the action of the operators $a(x), a^\dagger(x')$. The state vectors defined as

$$
|x_1, \ldots, x_n\rangle = \left(\prod_{i=1}^{n} a^\dagger(x_i)\right)|0\rangle, \qquad x_i \in \mathbf{R}^s \tag{A.3}
$$

satisfy the following orthogonality relation

$$
\langle x_1, \ldots, x_n | x'_1, \ldots, x'_m\rangle = \delta_{nm} \sum_{\sigma} \prod_{i=1}^{n} \delta(x_i - x'_{\sigma(i)}), \tag{A.4}
$$

where σ is a permutation of the set $\{1, \ldots, n\}$, and completeness relation

$$
\sum_{n} \frac{1}{n!} \int d^s x_1 \ldots d^s x_n |x_1, \ldots, x_n\rangle\langle x_1, \ldots, x_n| = I. \tag{A.5}
$$

The vectors $|x_1, \ldots, x_n\rangle$ form the basis of the coordinate representation. The Bose operators act on the basis vectors as follows

$$a(x)|x_1,\ldots,x_n\rangle = \sum_{i=1} \delta(x-x_i)|x_1,\ldots,\check{x}_i,\ldots,x_n\rangle,$$

$$\tag{A.6}$$

$$a^\dagger(x)|x_1,\ldots,x_n\rangle = |x_1,\ldots,x_n,x\rangle,$$

where the reversed hat over x_i denotes that this variable should be omitted from the set $\{x_1,\ldots,x_n\}$.

Appendix B. Functional coherent states representation

We now recall the basic properties of the functional coherent states. Consider the functional coherent states $|u\rangle$, where $u \in L^2(\mathbf{R}^s, d^s x)$ (the complex Hilbert space of square-integrable functions). The functional coherent states can be defined as the eigenvectors of the Bose annihilation operators

$$a(x)|u\rangle = u(x)|u\rangle. \tag{B.1}$$

The normalized functional coherent states can be defined as

$$|u\rangle = \exp\left(-\frac{1}{2}\int d^s x|u|^2\right)\exp\left(\int d^s x u(x) a^\dagger(x)\right)|0\rangle. \tag{B.2}$$

These states are not orthogonal. We find

$$\langle u|v\rangle = \exp\left(-\frac{1}{2}\int d^s x(|u|^2+|v|^2-2u^* v)\right). \tag{B.3}$$

The coherent states form the complete (overcomplete) set. The formal resolution of the identity can be written as

$$\int\limits_{\Omega^2} D^2 u|u\rangle\langle u| = I, \tag{B.4}$$

tempered distributions, $D^2u = D(\mathrm{Re}u)D(\mathrm{Im}u)$ and the symbol $\exp\left(-\int d^s x v^2\right) Dv$, where $v \in L^2_{\mathbf{R}}(\mathbf{R}^s, d^s x)$ (the real Hilbert space of square-integrable functions) designates the Gaussian measure on Ω.

The passage from the coordinate representation to the functional coherent states representation is given by

$$\langle x_1, \ldots, x_n | u \rangle = \left(\prod_{i=1}^{n} u(x_i) \right) \exp\left(-\frac{1}{2} \int d^s x |u|^2 \right). \tag{B.5}$$

Suppose now that we are given an arbitrary state $|\phi\rangle$. It follows immediately from (A.5) and (B.5) that the functional $\phi[u^*] = \langle u|\phi\rangle$ is of the form

$$\phi[u^*] = \tilde{\phi}[u^*] \exp\left(-\frac{1}{2} \int d^s x |u|^2 \right), \tag{B.6}$$

where the functional $\tilde{\phi}[u^*]$ is analytic.

An easy calculation based on (B.1), (B.2) and (B.6) shows that (B.6) can be written in the following abstract basis independent form

$$|\phi\rangle = \tilde{\phi}[a^\dagger]|0\rangle. \tag{B.7}$$

Taking into account (B.[illegible]) and (B.[illegible]) we find

$$\langle\phi|\psi\rangle = \int\limits_{\Omega^2} D^2u \exp(-\textstyle\int d^sx|u|^2)\tilde{\phi}^*[u^*]\tilde{\psi}[u^*]. \tag{B.8}$$

The representation (B.8) is the functional Bargmann representation. The Bose operators act in this representation as follows

$$\begin{aligned} a(x)\tilde{\phi}[u^*] &= \frac{\delta}{\delta u^*(x)}\tilde{\phi}[u^*], \\ \\ a^\dagger(x)\tilde{\phi}[u^*] &= u^*(x)\tilde{\phi}[u^*]. \end{aligned} \tag{B.9}$$

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